THE MATHEMATICS BEHIND MUSIC

MATHEMATICS BEHIND MUSICAL INSTRUMENT

VOL – 1

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THE MATHEMATICS BEHIND MUSIC

PREFACE

Music theory has no axiomatic foundation in modern mathematics, although some interesting work has recently been done in this direction (see the External Links), yet the basis of musical sound can be described mathematically (in acoustics) and exhibits "a remarkable array of number properties". Elements of music such as its form, rhythm and metre, the pitches of its notes and the tempo of its pulse can be related to the measurement of time and frequency, offering ready analogies in geometry.

The attempt to structure and communicate new ways of composing and hearing music has led to musical applications of set theory, abstract algebra and number theory. Some composers have incorporated the golden ratio and Fibonacci numbers into their work.

Though ancient Chinese, Indians, Egyptians and Mesopotamians are known to have studied the mathematical principles of sound, the Pythagoreans (in particular Philolaus and Archytas) of ancient Greece were the first researchers known to have investigated the expression of musical scales in terms of numerical ratios, particularly the ratios of small integers. Their central doctrine was that "all nature consists of harmony arising out of numbers".

From the time of Plato, harmony was considered a fundamental branch of physics, now known as musical acoustics. Early Indian and Chinese theorists show similar approaches: all sought to show that the mathematical laws of harmonics and rhythms were fundamental not only to our understanding of the world but to human well-being. Confucius, like Pythagoras, regarded the small numbers 1,2,3,4 as the source of all perfection.

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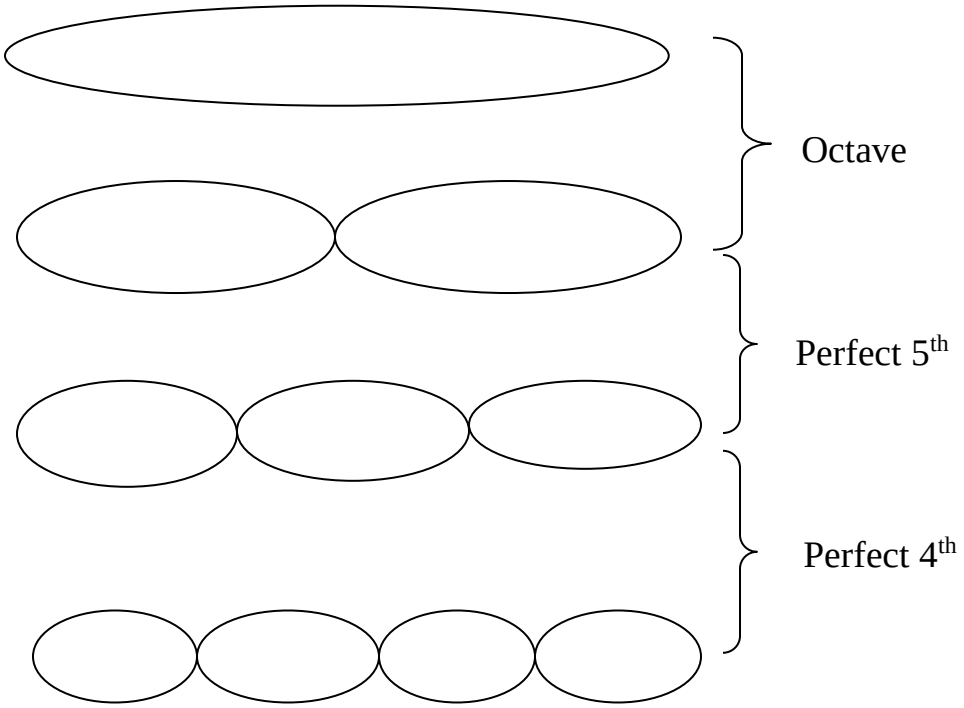
MATHEMATICS BEHIND PIANO KEYS-1

Arrangement of piano scale and tuning

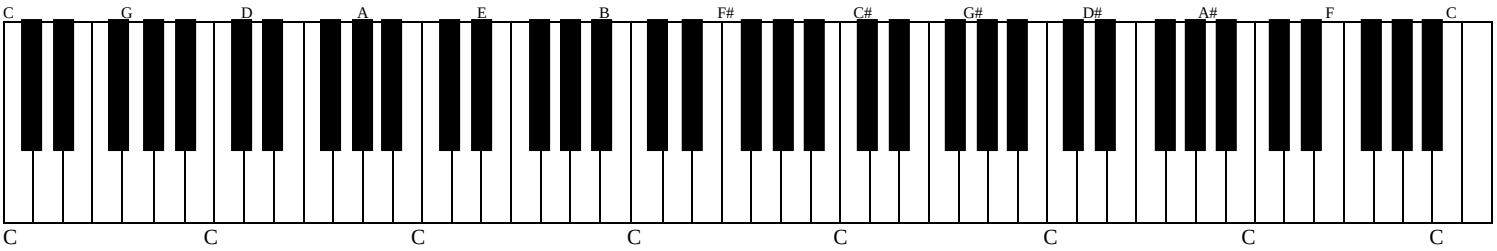
Pythagorean Tuning:



Ratio for Octave – 2:1
Ratio for Perfect fifth – 3:2 (or) 1.5:1
Ratio for Perfect fourth – 4:3 (or) 1.333…:1
Ratio for Major third – 5:4 (or) 1.25:1
Ratio for Minor third – 6:5 (or) 1.2:1

Mathematical Proof for Piano to be in tune:

Can be Mathematically prove for Piano to be in tune????



Assume that ,

$$2^7 = \left(\frac{3}{2}\right)^{12}$$

$$\Rightarrow 2^7 . 2^{12} = 3^{12}$$
$$\Rightarrow 2^{19} = 3^{12}$$
$$\Rightarrow \text{Even} = \text{Odd}$$

$\Rightarrow\Leftarrow$ Mathematically prove for Piano to be in tune while using Pythagorean ratio

We can say ,

$$2^7 \cong \left(\frac{3}{2}\right)^{12}$$

Equal Temperament Scale Tuning:

This is one of the application using Fibonacci Sequence. In Piano keys, there are 12 notes in an octave. Each notes have same ratio called "r". The ratio "r" is called Golden Ratio.

Finding 'r' using Mathematical way:

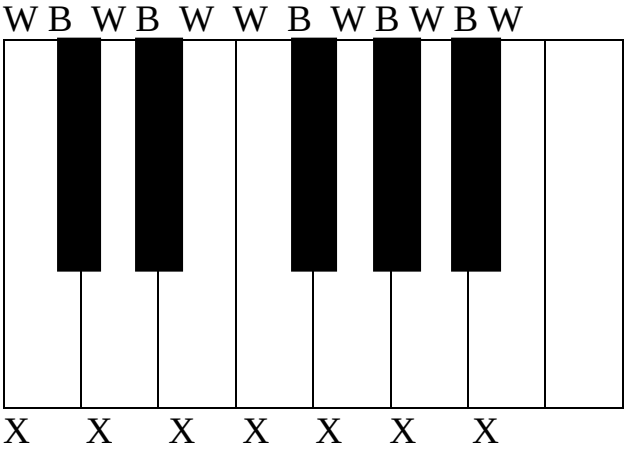
The notes in an octave are C, C#, D, D#, E, F, F#, G, G#, A, A#, B and the corresponding frequencies are f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11} and the equal ratio is 'r'.
For example: $f_1=rf_0$, $f_2=rf_1=r^2f_0$. So, $f_{12}=r^{12}f_0$. The ratio of an octave is 2:1 …..(1)
and the frequency is r^{12}:1 …..(2)
By (1) and (2),

$$r^{12} = 2$$
$$r = \sqrt[12]{2} = 1.05946$$

MATHEMATICS BEHIND PIANO KEYS -2

Blind Person Plays Piano

In Harmonium keys, the width of both white and black keys are equal. So, even the blind person can play Harmonium. But, now in electric piano, they have different width in black and white keys. Using mathematical way, we can solve the problem.



In the above picture, the X denotes the width of white keys in front, W denotes the width of white keys surrounded by corresponding black keys, and B denotes the width of black keys. We get,

$$3X=3W+2B \quad \ldots\ldots(1)$$
$$4X=4W+3B \quad \ldots\ldots(2)$$

Solving (1) and (2), we get

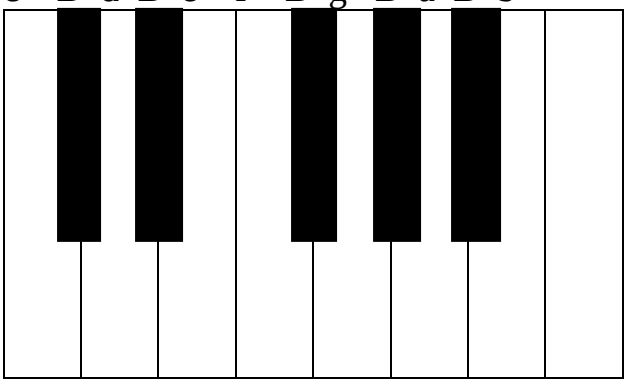
$B=3/2(X-W)$ $B=4/3(X-W)$

$\Rightarrow$ Impossible or $B=0$

Case-1:

If c=e=f=b, (which is surrounded by 1 white key and 1 black key) and d=g=a (which is surrounded by 2 black keys), we get

c B d B e f B g B a B b



X X X X X X X

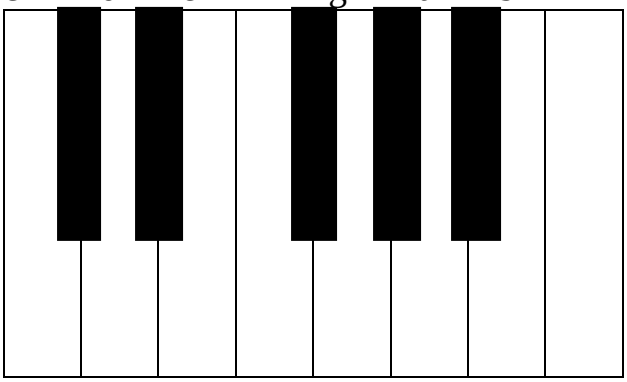
3X-2B=2c+d …….(3)
4X-3B=c+2d …….(4)

Solving (3) and (4), we get
c=X-B/2 d=X-B
So, d-c=B/2, which is larger to fingering.

Case-2:

If c=d=e, nd f=g=a=b

c B d B e f B g B a B b



X X X X X X X

3X-2B=3c ⇒ c=X-(2/3)B
4X-3B=4f ⇒ f=X-(3/4)B

f-c=B/12 < B/2, which is better for fingering

So that way only, the electric piano has different width according to case-2 and even blind person can play piano easily.

MATHEMATICS BEHIND GUITAR STRUCTURE

Natural Notes:

Our natural tuning is 440Hz (the frequency of A). So, it starts from A⇢A'

A	A#	B	C	C#	D	D#	E	F	F#	G	G#
0	1	2	3	4	5	6	7	8	9	10	11

Open String on a Guitar:

There are 6 strings in guitar. The notes in the string of guitar are as follows:

7	E	Bottom
0	A	Next to E
5	D	Next to A
10	G	Next to D
2	B	Next to G
7	E	Top (Next to B)

There are about 24 frets in fretboard. The zero fret is called nut which is in the top of the guitar. The last fret and continues till the end of string is called bridge which is in the bottom of the guitar.

Mathematical Problem for Guitar:

It is easy to learn guitar in mathematical way using mathematical problem

If we take the scale G# (ie. 11) and the fret no is 15 then,

$$11+15=26>11$$

So, we can subtract by 12, we get

$$26-12=14>11$$

Repeating the above process, we get

$$14-12=2 \Rightarrow \text{The value of scale B}$$

Relationship between Length of string and Frequency:

We know that in Pythagorean tuning, the ratio of octave is 2:1,
Let, l_1 and l_2 be the length of the string. (i.e. $l_2=(1/2)l_1$) and the corresponding frequencies f_1 and f_2 of l_1 and l_2 are doubled
So,

$$f \propto \frac{1}{L} \qquad \ldots.(1)$$

i.e. The frequency is inversely proportional to the length of string.

Relationship between frequency and the tension:

In the guitar, there are 6 pegs for tuning the strings. The pegs are used to adjust the string to correct the perfect tuning. While adjusting the string using pegs, the tension is produced.
So,

$$f \propto \sqrt{T} \qquad \ldots.(2)$$

i.e. The frequency is directly proportional to the square root of the tension of the string.

Relationship between Density and Frequency:

Let, μ be the density of the string
So,

$$f \propto \frac{1}{\sqrt{\mu}} \qquad \ldots.(3)$$

i.e. The frequency is inversely proportional to the square root of the density of the string.

Frequency formula for Guitar:

By (1), (2), & (3), we get,

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

where, L= Length of String
T= Tension of string
μ=Density of String

Mathematical formula for frets:

There are 2 rules by using frets,

✓ Rule of 18:

The actual value of Rule of 18 is 17.817. This rule is to find the length of string from the nut to the bridge.

$$\text{Distance of first fret} = \frac{\text{Length of string}}{17.817}$$

$$\text{Distance of second fret} = \frac{\text{Length of string} - \text{Distance of first fret}}{17.817}$$

Example 1:
If the string length is 658mm. Find the distance of 1st and 2nd fret.

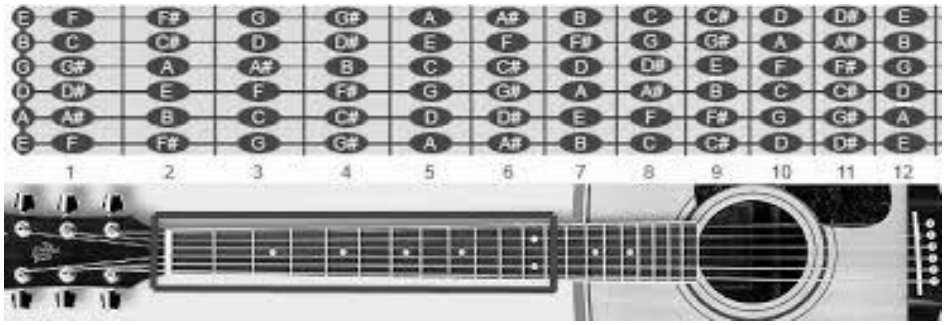
Solution:
Given,
Length of the sting =658mm

$$\text{Distance of first fret} = \frac{658}{17.817} = 36.931\text{mm}$$

$$\text{Distance of second fret} = \frac{658 - 36.931}{17.817} = \frac{621.069}{17.817} = 34.858\text{mm}$$

✓ Equal Temperament:

After the Pythagorean ratio, they took the equal temperament scale tuning (the ratio of frequency between 2 notes is equal. The ratio is $\sqrt[12]{2}$)



$$\text{Distance } (x_i) = \frac{s}{2^{x_i/12}} \text{ where } i = 1,2,\ldots,n$$

Example 2:
Continue to Example 1, using the above formula, find the distance of 1st and 2nd fret.

Solution:

$$\text{Distance}(1) = \frac{658}{2^{1/12}} = \frac{658}{1.05946} = 621.069\text{mm}$$

$$\text{Distance of first fret} = 658 - 621.069 = 36.9307\text{mm}$$

$$\text{Distance}\ (2) = \frac{658}{2^{2/12}} = \frac{658}{1.12246} = 586.2113\text{mm}$$

$$\text{Distance of Second fret} = 658 - 586.2113 - 36.9307 = 34.8579\text{mm}$$

MATHEMATICS BEHIND VIOLIN

Strings in a Violin:

There are 4 strings in a violin

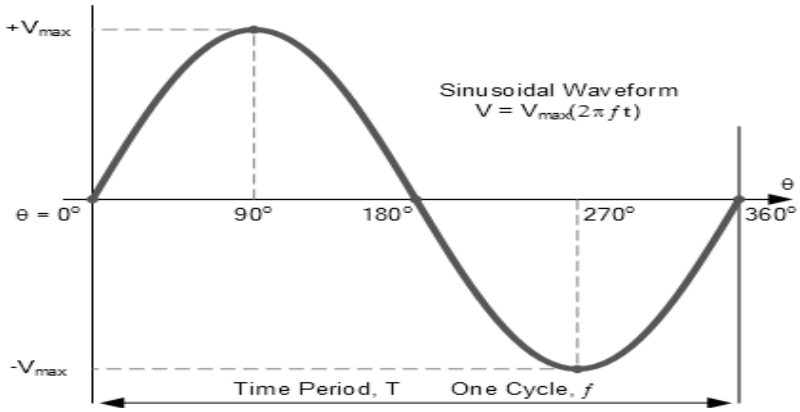
G	Bottom
D	Next to G
A	Next to D
E	Top (Next to A)

Harmonics:

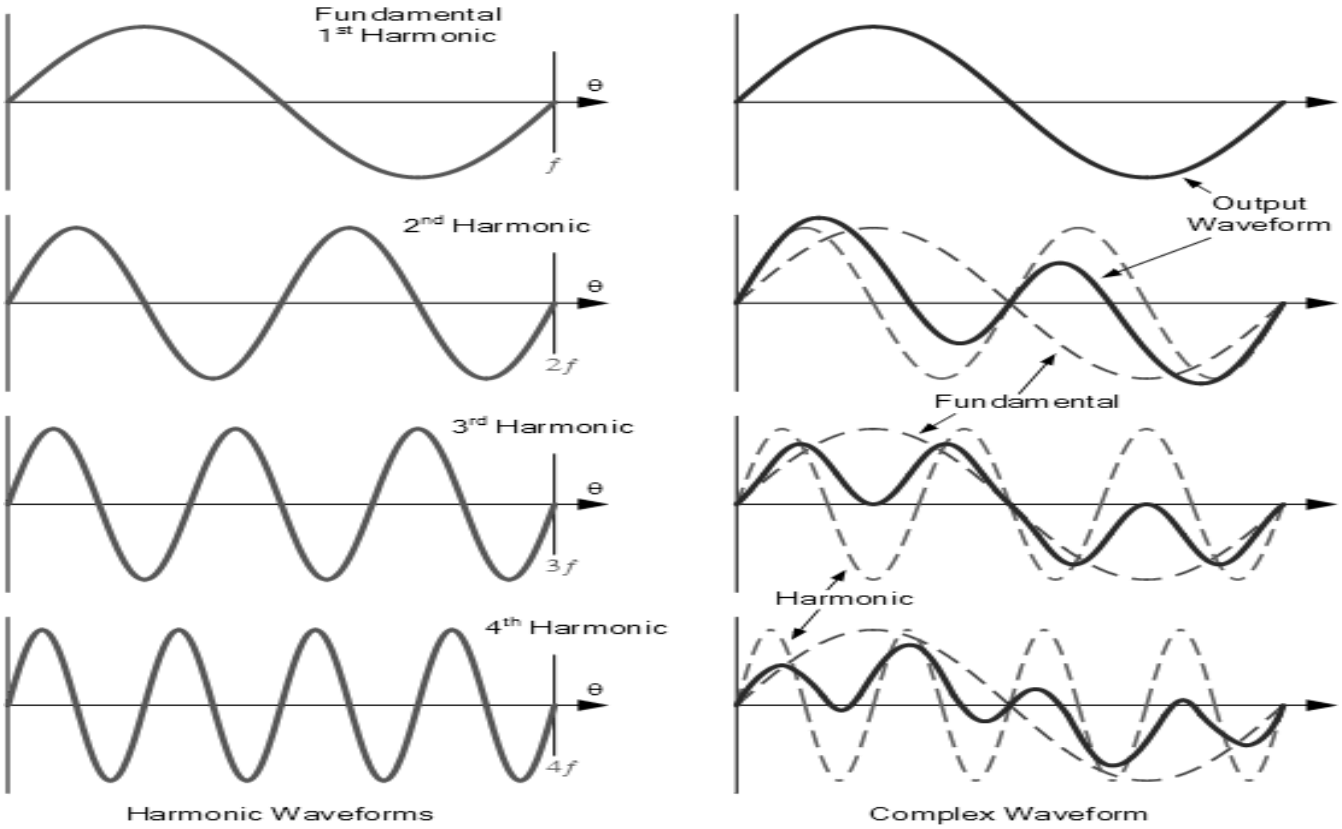
No sounds have a pure tone. Sounds are made up of harmonics. There are many different harmonics in sound.

First Harmonic:

The first harmonic is also called fundamental frequency.



The following harmonics are given below:



Frequency formula for a Violin:

We know that, the frequency formula for guitar is

$$f = \frac{\sqrt{T/\mu}}{2L}$$

But density $\mu = m/L$

where, m=Mass of the string

L=Length of the string

So,

$$f_1 = \frac{1}{2L}\sqrt{\frac{T}{m/L}}$$

Similarly,

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{m/L}}$$

Where, T= Tension of the string produced by Pegs.

Helmholtz Motion:

In our naked eye, while bowing a violin, the movements of string looks like a parabolic form. The German physician and scientist, Hermann Von Helmholtz (1821-1894) sees and concentrates that the movement of strings is in v-shaped. The two straight line meets the corner while is in sticking and slipping. This corner is called Helmholtz corner.

The bow movement induces the stick-slip effect.

This creates the Sinusoidal function

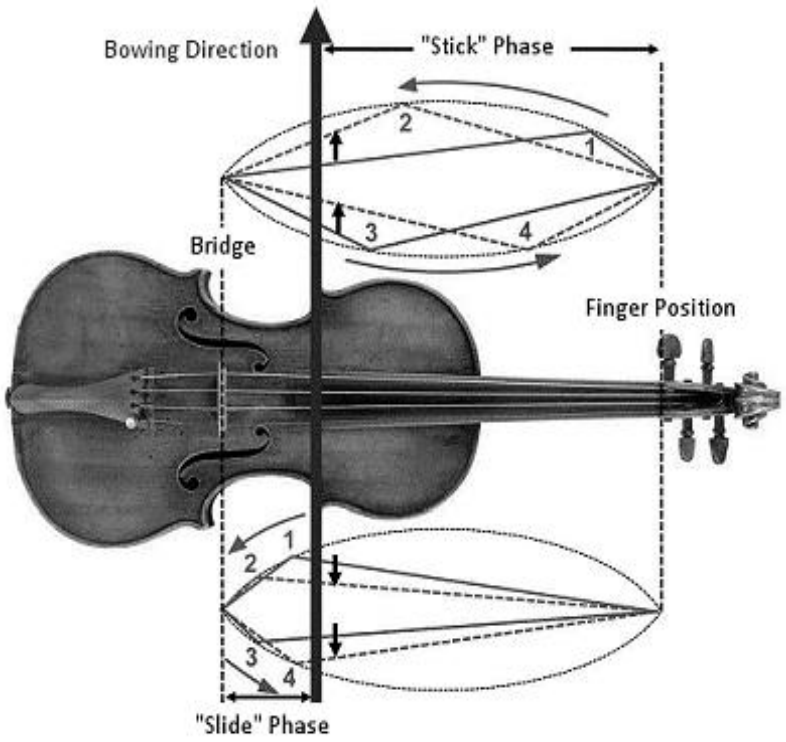
$$y = A\sin\big(B(x - C)\big) + D$$

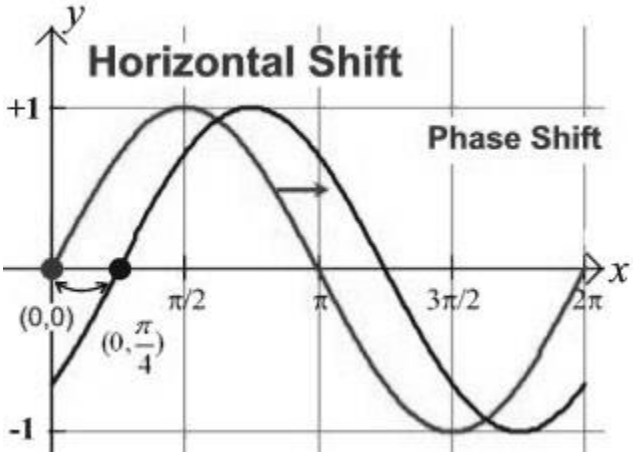
where, A= Amplitude

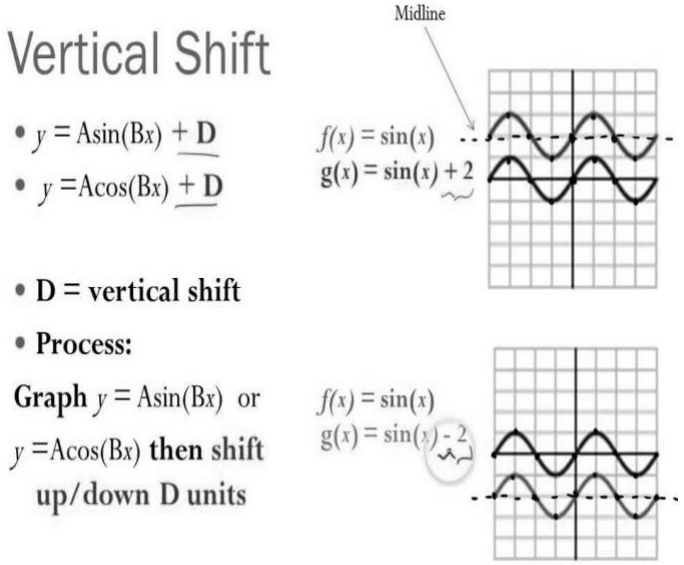
B= Cycle ($\frac{2\pi}{B}$ ⇢ Period)

C= Phase Shift

D= Vertical Shift







MATHEMATICS BEHIND PAN FLUTE

Length of Pipe:

Pan flute is made up of bamboo pipes containing approximately 18 pipes. The length of pipe can be measured by using frequency. We know that, the frequency is inversely proportional to the wavelength. The relationship between the frequency and the length of pipe is given as follows:

$$\text{Length of pipe} = \frac{\text{Speed of sound}}{2 * \text{Frequency}} + \text{Radius}$$

Where, Speed of sound is approximately equals to 340 m/s or 13500 inch/s.

Example 1:
If the frequency is 204 Hz and the diameter is 12 cm. Find the length of pipe.

Solution:
Given,
Frequency = 204 Hz
Diameter = 12 cm = 0.12 m
Radius = 0.06 m

$$\text{Length of Pipe} = \frac{340}{2 * 204} + 0.06 = 0.89 \text{ m} = 89 \text{ cm}$$

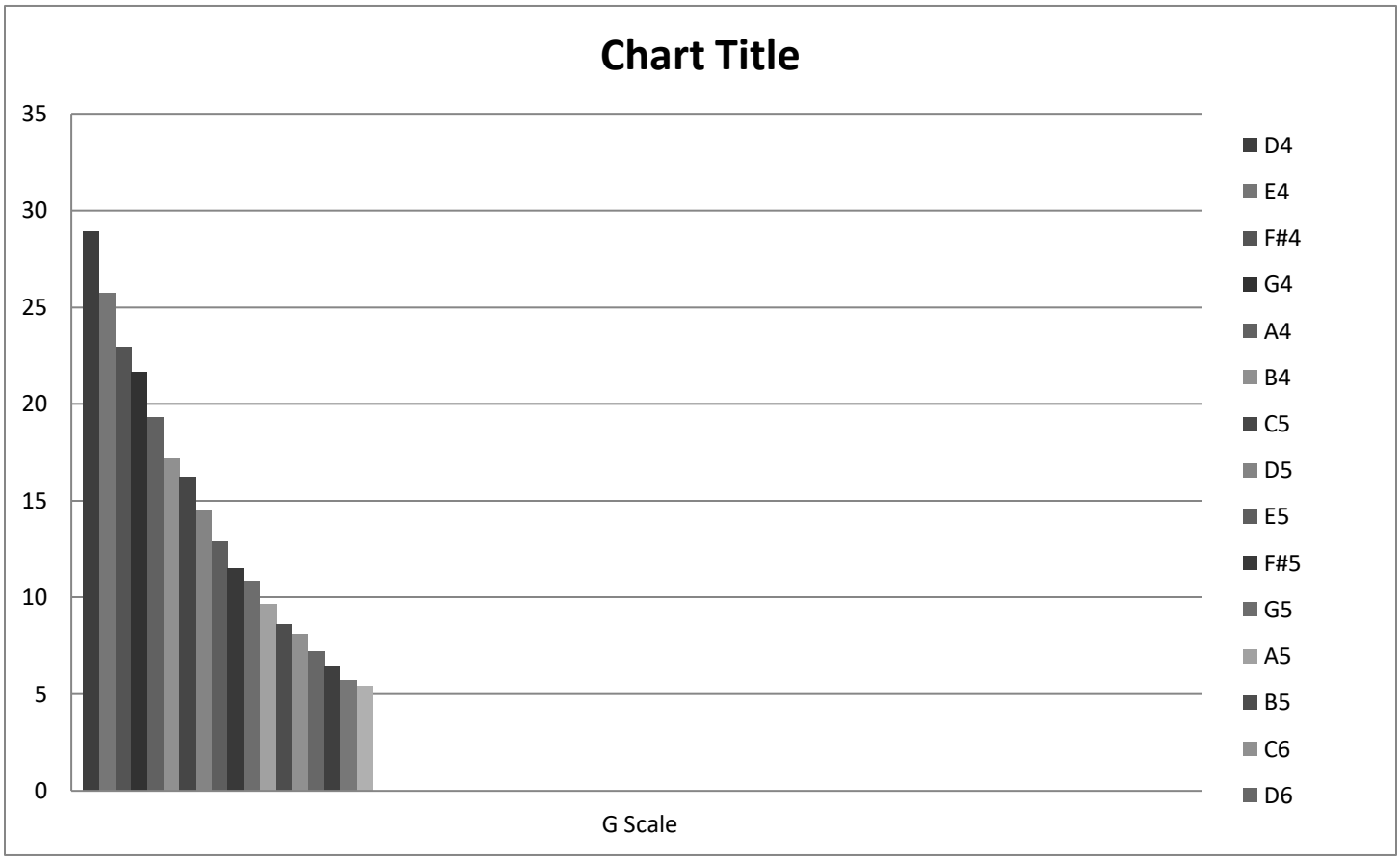
Name of Instrument:

The name "Pan flute" is reference to the Greek God "Pan". In ancient Greek religion and mythology, **Pan** is the god of the wild, shepherds and flocks, nature of mountain wilds, rustic music and impromptus, and companion of the nymphs. He has the hindquarters, legs, and horns of a goat, in the same manner as a faun or satyr. With his homeland in rustic Arcadia, he is also recognized as the god of fields, groves, wooded glens and often affiliated with sex; because of this, Pan is connected to fertility and the season of spring. The ancient Greeks also considered Pan to be the god of theatrical criticism. The word panic ultimately derives from the god's name.

In Roman religion and myth, Pan's counterpart was Faunus, a nature god who was the father of Bona Dea, sometimes identified as Fauna; he was also closely associated with Sylvanus, due to their similar relationships with woodlands. In the 18th and 19th centuries, Pan became a significant figure in the Romantic movement of western Europe and also in the 20th-century Neopagan movement.

Pipes of Pan Flute:

We know that, there are approximately equals to 18 pipes in one scale. For example, in G scale,



In the above figure, there are 18 pipes in G scale which starts from D4 to G6.

Length of tube in Pan Flute using Frequency:

In the above figure, the length of D4 is 29 cm and the length of D5 is 14.5 cm. The corresponding frequencies of D4 and D5 are 294 Hz and 588 Hz.
So, we can conclude that for the 1st frequency,

$$L = \frac{c}{4f_1}$$

Where, c = Speed of sound
For the n^{th} frequency,

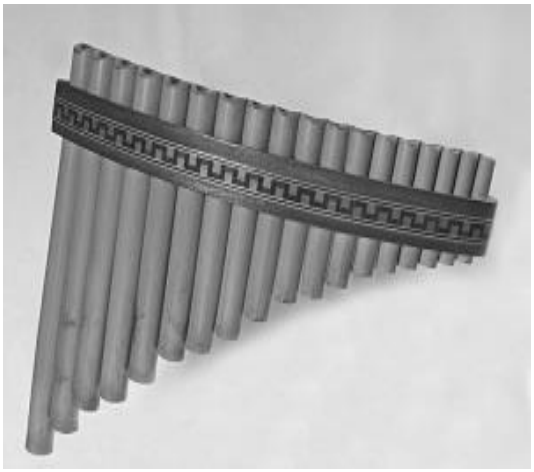
$$L = \frac{c}{4f_n}$$

But in Equal Temperament tuning,

$$f_n = 2^{n/12} f_1$$

So,

$$L = \frac{c}{4 * 2^{n/12} * f_1}$$



Example 2:
If the frequency is 440 Hz and the number of note is 8. Find the length of tube

Solution:

Given,

$f_1 = 440$ Hz

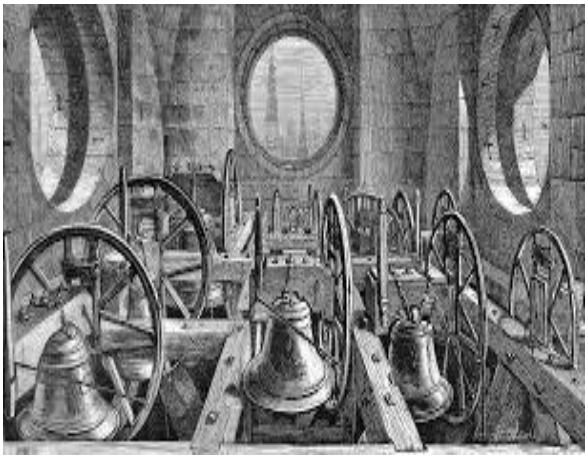
$n = 8$

$$L = \frac{340}{4 * 2^{8/12} * 440} = \frac{340}{2793.83} = 0.1217 \text{ m} = 12.17 \text{ cm}$$

MATHEMATICS BEHIND CAMPANOLOGY

Campanology- Definition:

Campanology is defined as the study of Bells.



Metals using Bells:

The bells are manufactured by using alloy. The metals which are manufacturing bells is given as follows:

Copper	80%
Tin	20%

Arrangements of Bells:

The bells are arranged from highest bell (called treble) to lowest bell (called tenor).

Carillons:

Collection of tuned bells is called carillons.
Carillon music is typically written on two staves. Notes written in the bass clef are generally played by the feet. Notes written in the treble clef are played with the hands. Pedals range from the lowest note (the bourdon) and may continue up to two and half octaves. In the North American Standard keyboard, all notes can be played on the manual.
Because of the acoustic peculiarities of a carillon bell (the prominence of the minor third, and the lack of damping of sound), music written for other instruments needs to be arranged specifically for the carillon.

The combination of carillon and other instruments, while possible, is generally not a happy marriage. The carillon is generally far too loud to perform with most other concert instruments. The great exceptions to this are some late twentieth and early twenty-first century compositions involving electronic media and carillon. In these compositions, sound amplification is able to match the extreme dynamic range of the carillon and, in the case of sensitive composers, even the most delicate effects are possible. Brass music is often heard together with a (traveling) carillon.

Chime:

Carillon like instrument with fewer than 23 bells is called chime.

The first bell chime was created in 1487. Before 1900, chime bells typically lacked dynamic variation and were not harmonically tuned. Since then chime bells produced in Belgium, the Netherlands, England, and America have tuning to produce fully harmonized music. Some towers in England normally hung for full circle change ringing can be chimed when mouth downwards by an Ellacombe apparatus.

American chimes usually have one to one and a half diatonic octaves. Some chimes are automated.

Different types of bell tuning:

After manufacturing, they check the proper tunes depend upon the thickness and the diameter of bells.

$$\therefore f = \frac{Ch}{D^2}$$

Where, C = constant

h = Thickness

D = Diameter

There are 5 tuning depend upon the constants

Strike Note:

If a bell is a part of a set to be rung or played together, then the initial dominant perceived sound called the strike note.

Hum Note:
An octave below the strike note is called Hum note.

Tierce:
Minor third above the strike note is called Tierce.

Quint:
Perfect fifth above the strike note is called quint.

Nominal:
An octave above the strike note is called nominal.

Concepts using Bell Ringing:
The concepts while arranging the bell ringing is given below:

- Group theory
- Graph theory

Rule Name in 12 Bells:
We know that there are more than 3 bells in one tower in churches. Among the bell ringing, there are certain rules while ringing bells. The rules are given as follows:

No. of Bells	How many ways	Approximate Duration	Rule Name
3	3!	15 sec	Single
4	4!	1 min	Minimus
5	5!	5 min	Doubles
6	6!	30 min	Minor
7	7!	3 hrs	Triples
8	8!	24 hrs	Major
9	9!	9 dys	Caters
10	10!	3 mon	Royal
11	11!	3 yrs	Cinques
12	12!	36 yrs	Maximus

Problem for arranging 3 Bells:

This problem can be applied using one of the Group theory which is called Permutation group by using transposition according to the following rules:

- ✓ Numbers cannot be interchange more than one place at a time.
- ✓ No repetitions should be in round.

For 3 bells named as 1 2 3

Let a =(1 2) and b =(2 3)
The problem is given as follows,

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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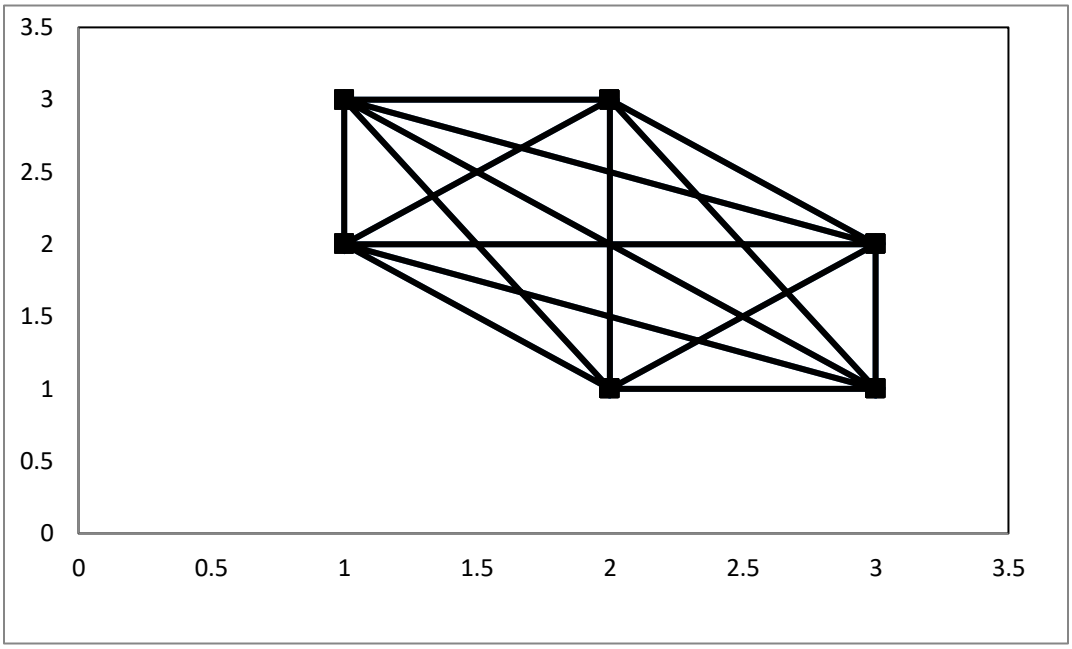
$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

The arrangements of bell ringing in 3 bells are:

1 2 3 ⇢ 2 1 3 ⇢ 2 3 1 ⇢ 3 2 1 ⇢ 3 1 2 ⇢ 1 3 2 ⇢ 1 2 3

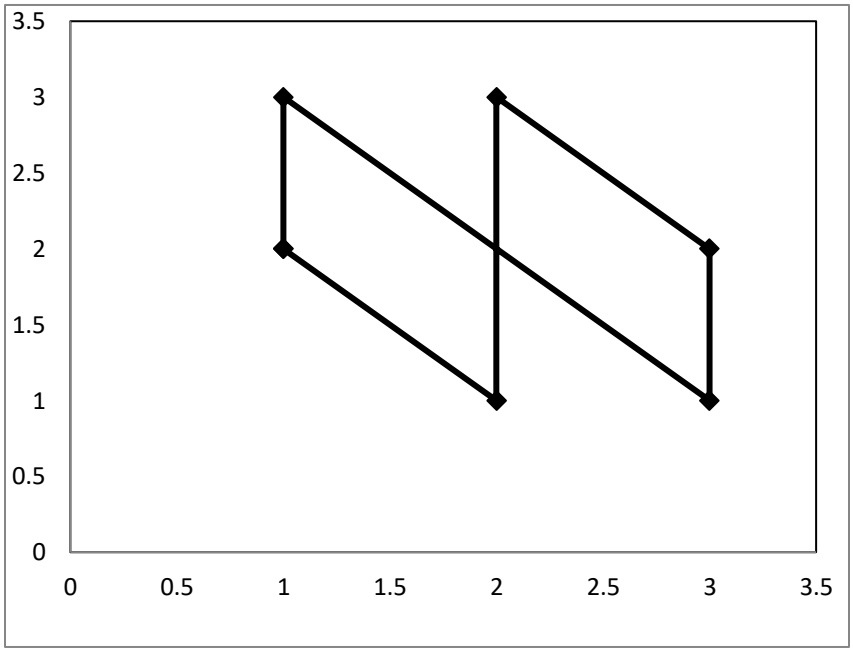
Graph out Bell Ringing for 3 Bells:

Each arrangement could be taken as vertex



Hamiltonian Cycle:

A Hamiltonian path or traceable path is a path that visits each vertex of the graph exactly once. A graph that contains a Hamiltonian path is called a traceable graph. A graph is Hamiltonian-connected if for every pair of vertices there is a Hamiltonian path between the two vertices.



A Hamiltonian cycle, Hamiltonian circuit, vertex tour or graph cycle is a cycle that visits each vertex exactly once. A graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

Similar notions may be defined for directed graphs, where each edge (arc) of a path or cycle can only be traced in a single direction (i.e., the vertices are connected with arrows and the edges traced "tail-to-head").

A Hamiltonian decomposition is an edge decomposition of a graph into Hamiltonian circuits.

A Hamilton maze is a type of logic puzzle in which the goal is to find the unique Hamiltonian cycle in a given graph.